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SOLUTION OF ST. VENANT'S TORSION PROBLEM BY POWER SERIES

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Abstract—It is shown how St. Venant's problem of torsion of elastic bars can be formulated so as to fit readily into the procedure for deriving approximate equations of motion of bars by expansion of the displacements in a double series of powers of the transverse coordinates. Applications are exhibited for bars with elliptic, equilateral triangular and rectangular cross-sections.

INTRODUCTION

Approximate equations of motion of elastic bars, obtained by expansion of displacements in a double power series of the transverse coordinates and subsequent truncation, are not as fully developed as those for plates. This is especially true for motions involving torsion, in which the warping of the cross-section introduces a complication which is present no matter how low the frequency and, in fact, persists even in the equilibrium state. In uniform extension or flexure of cylindrical or prismatic bars of isotropic materials, plane sections remain plane regardless of their contour, so that only linear terms in the power series expansion are necessary. In uniform torsion, on the other hand, plane sections remain plane only for the circular section. For all other sections axial warping occurs—and the character of the warping is different for different sections. There are exact solutions of the St. Venant torsion problem for a great many cross-sectional shapes; but they are not as easily incorporated in equations of motion obtained by power series expansion as they would be if they themselves had been similarly obtained.

In a previous paper[1] it was shown how the power series formulation of the problem of torsion of isotropic bars, by Bleustein and Stanley[2], can be extended to accommodate appropriate warping and produce suitable torsional rigidity without the introduction of correction factors: applied to moments of area of the section or otherwise. In the present paper, the direct connection of the series treatment with St. Venant's method of solution is demonstrated. The torsion function, instead of being governed by Laplace's equation, is expressed as a double power series, the coefficients of which are determined by simultaneous linear algebraic equations, deduced from a variational principle and equal in number to that of the terms of the series retained to represent the axial displacement. These terms are not necessarily all the early terms of the series up to a certain order, but are only a few isolated terms, judiciously chosen. For example, if the St. Venant torsion function is a polynomial (the simplest are for the elliptic and equilateral triangular sections) those are the only terms retained and the resulting torsional rigidity from the power series solution is exact. Otherwise, as in the case of the rectangular section for which the torsion function is an infinite series of transcendental functions, a limited selection of terms in the power series is retained as an approximation. Thus, for the square, a two-term approximation gives the torsional rigidity with an error of about 0·1 per cent. For other rectangular sections, with side-ratio k, say, a four-term approximation gives torsional rigidities with errors increasing from the 0·1 per cent for $k = 1$ to about 3 per cent around $k = 10$ and then diminishing to zero as k approaches infinity.

The criterion of acceptability of the choice of the terms to be retained in the power series is the torsional rigidity, as that determines the torque which balances the inertia in the equation of torsional motion and thus controls the long-wave-speed which, along with the length of the bar and the end-conditions, fixes the lower natural frequencies of torsional vibration.

The procedure described here can readily be applied to vibrations of anisotropic bars in which the torsional motion is coupled with extensional and flexural motions[3].

DERIVATION OF EQUATIONS

The St. Venant problem of torsion of cylindrical or prismatic bars of isotropic material may be stated as follows (Ref. [4], p. 311). With the generators of the surface parallel to the axis x_3 of a rectangular coordinate system x_i, $i = 1, 2, 3$, the displacement components are taken to be

$$u_1 = -\tau x_2 x_3,\ u_2 = \tau x_3 x_1,\ u_3 = \tau\varphi, \tag{1}$$

where τ is the twist and φ is a function of x_1 and x_2 satisfying Laplace's equation

$$\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} = 0 \tag{2}$$

in the interior of the bar and

$$\frac{\partial\varphi}{\partial\nu} = x_2\cos(x_1, \nu) - x_1\cos(x_2, \nu) \tag{3}$$

on the cylindrical or prismatic surface, to which $d\nu$ is the element of the outward drawn normal. The torsional rigidity is the ratio of the torque M to the twist and is given by

$$C = \frac{M}{\tau} = \mu\int\int_A \left(x_1^2 + x_2^2 + x_1\frac{\partial\varphi}{\partial x_2} - x_2\frac{\partial\varphi}{\partial x_1}\right) dx_1\, dx_2, \tag{4}$$

where μ is the modulus of rigidity and the integration is over the area of the section.

Instead of seeking solutions of (2) satisfying (3), we take φ as a double series of powers of the coordinates x_1 and x_2:

$$\varphi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_1^m x_2^n u_3^{m,n}, \tag{5}$$

in which the $u_3^{m,n}$ are constants. The displacements (1) may then be written as

$$u_1 = \tau x_2 u_1^{0,1},\ u_2 = \tau x_1 u_2^{1,0},\ u_3 = \tau\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_1^m x_2^n u_3^{m,n}, \tag{6}$$

where $u_1^{0,1} = -x_3$, $u_2^{1,0} = x_3$; so that the components of strain:

$$S_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}),$$

become $S_{11} = S_{22} = S_{33} = S_{12} = 0$ and

$$S_{3\alpha} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x_1{}^m x_2{}^n S_{3\alpha}^{m,n},\ \alpha = 1, 2, \tag{7}$$

where

$$S_{31}^{0,1} = \tfrac{1}{2}\tau(u_3{}^{1,1} - 1),\ \ S_{32}^{1,0} = \tfrac{1}{2}\tau(u_3{}^{1,1} + 1) \tag{8}$$

and, otherwise,

$$S_{31}^{m,n} = \tfrac{1}{2}\tau(m+1)u_3{}^{m+1,n},\ \ S_{32}^{m,n} = \tfrac{1}{2}\tau(n+1)u_3{}^{m,n+1}. \tag{9}$$

The components of stress are $T_{11} = T_{22} = T_{33} = T_{12} = 0$ and

$$T_{3\alpha} = 2\mu S_{3\alpha} = 2\mu \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x_1{}^m x_2{}^n S_{3\alpha},\ \ \alpha = 1, 2. \tag{10}$$

These components are substituted in the variational equation

$$\int_A \int_{-l}^{l} T_{ij,i}\delta u_j \, dx_3 \, dA + \int_S (t_j - \nu_i T_{ij})\delta u_j \, dS = 0, \tag{11}$$

where $\int_A (\,)\,dA$ is the integral over the area of the cross-section, $2l$ is the length of the bar, $\int_S (\,)\,dS$ is the integral over the surface of the bar, including the end faces, and t_j is the surface traction. On substituting (6)–(10) in (11), and performing the integrations where possible, we find

$$j = 1:\ \int_{-l}^{l} F_1{}^{0,1}\delta u_1{}^{0,1}\, dx_3 + [t_1{}^{0,1} - T_{31}^{0,1}]_{-l}^{l}\delta u_1{}^{0,1} = 0, \tag{12}$$

$$j = 2:\ \int_{-l}^{l} F_2{}^{1,0}\delta u_2{}^{1,0}\, dx_3 + [t_2{}^{1,0} - T_{32}^{1,0}]_{-l}^{l}\delta u_2{}^{1,0} = 0, \tag{13}$$

$$j = 3:\ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \int_{-l}^{l} (mT_{31}^{m-1,n} + nT_{32}^{m,n-1} - F_3{}^{m,n})\, dx_3 - [t_3{}^{m,n}]_{-l}^{l} \right\} \delta u_3{}^{m,n} = 0, \tag{14}$$

where

$$T_{3\alpha}^{m,n} = \int_A T_{3\alpha} x_1{}^m x_2{}^n \, dA,\ \ F_j{}^{m,n} = \oint_C t_j x_1{}^m x_2{}^n \, ds,\ \ t_j{}^{m,n} = \int_A t_j x_1{}^m x_2{}^n \, dA \tag{15}$$

and $\oint_C (\,)\,dS$ is the line integral around the bounding curve of the section.

The condition that the cylindrical surface be free of traction requires $F_j{}^{m,n} = 0$. Then, from (14), we find the equilibrium equations, one for each $u_3{}^{m,n}$,

$$mT_{31}^{m-1,n} + nT_{32}^{m,n-1} = 0 \tag{16}$$

and the requirement $[t_3{}^{m,n}]_{-l}^{l} = 0$, i.e. there can be no tension applied to the bar. The torque,

$t_2^{1,0} - t_1^{0,1}$, applied to the end faces is, from (12) and (13),

$$M = T_{32}^{1,0} - T_{31}^{0,1} = C\tau. \tag{17}$$

From the first of (15) and (7)–(10), we find

$$T_{31}^{m,n} = \mu\tau\left[-I^{m,n+1} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(p+1)I^{m+p,n+q}u_3^{p+1,q}\right], \tag{18}$$

$$T_{32}^{m,n} = \mu\tau\left[I^{m+1,n} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(q+1)I^{m+p,n+q}u_3^{p,q+1}\right], \tag{19}$$

where

$$I^{m+p,n+q} = \int_A x_1^{m+p}x_2^{n+q}\,\mathrm{d}A. \tag{20}$$

Upon substituting (18) and (19) in (16) and (17), we find the equilibrium equations

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[m(p+1)I^{m+p-1,n+q}u_3^{p+1,q} + n(q+1)I^{m+p,n+q-1}u_3^{p,q+1}] = mI^{m-1,n+1} - nI^{m+1,n-1}$$

and the torsional rigidity

$$C = \mu\left\{I^{0,2} + I^{2,0} - \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[(p+1)I^{p,q+1}u_3^{p+1,q} - (q+1)I^{p+1,q}u_3^{p,q+1}]\right\}. \tag{21}$$

ELLIPSE

The St. Venant torsion function for the bar of elliptic cross-section with semi-principal axes a and b along x_1 and x_2, respectively, is

$$\varphi = a^2b^2x_1x_2/(a^4 - b^4).$$

Hence, in the power series for u_3, we retain only $u_3^{1,1}$. The equilibrium equations (16) then reduce to the single equation for $m = 1$, $n = 1$:

$$T_{31}^{0,1} + T_{32}^{1,0} = 0, \tag{22}$$

where, from (18) and (19),

$$T_{31}^{0,1} = \mu\tau I^{0,2}(u_3^{1,1} - 1),\quad T_{32}^{1,0} = \mu\tau I^{2,0}(u_3^{1,1} + 1).$$

Upon substituting these values in (22) and solving for $u_3^{1,1}$, we find

$$u_3^{1,1} = (I^{0,2} - I^{2,0})/(I^{0,2} + I^{2,0})$$

and this, inserted in (21), yields the torsional rigidity

$$C = 4\mu I^{0,2}I^{2,0}/(I^{0,2} + I^{2,0}).$$

From (20) we have, for the ellipse,

$$I^{0,2} = \pi ab^3/4,\ I^{2,0} = \pi a^3 b/4.$$

Hence,

$$C = \pi\mu a^3 b^3/(a^2 + b^2),$$

which is the St. Venant torsional rigidity (Ref. [4], p. 316).

EQUILATERAL TRIANGLE

If the cross-section is an equilateral triangle with sides given by the equation (Ref. [4], p. 319)

$$(x_1 - a)(x_1 - x_2\sqrt{3} + 2a)(x_1 + x_2\sqrt{3} + 2a) = 0,$$

the torsion function φ is

$$\varphi = (3x_1^2 x_2 - x_2^3)/6a. \tag{23}$$

Hence, in the power series we retain only $u_3^{2,1}$ and $u_3^{0,3}$; so that the equilibrium equations (16) become, with $m = 2$, $n = 1$ and $m = 0$, $n = 3$:

$$2T_{31}^{1,1} + T_{32}^{2,0} = 0,\ T_{31}^{0,2} = 0.$$

From (18) and (19),

$$T_{31}^{0,1} = \mu\tau(-I^{0,2} + 2I^{1,2}u_3^{2,1}),\ T_{32}^{1,0} = \mu\tau(I^{2,0} + I^{3,0}u_3^{2,1} + 3I^{1,2}u_3^{0,3}),$$

$$T_{31}^{1,1} = \mu\tau(-I^{0,2} + 2I^{2,2}u_3^{2,1}),\ T_{32}^{2,0} = \mu\tau(I^{3,0} + I^{4,0}u_3^{2,1} + 3I^{2,2}u_3^{0,3}),$$

$$T_{32}^{0,2} = \mu\tau(I^{1,2} + I^{2,2}u_3^{2,1} + 3I^{0,4}u_3^{0,3}).$$

The equilibrium equations then become a pair of simultaneous equations for $u_3^{2,1}$ and $u_3^{0,3}$:

$$(4I^{2,2} + I^{4,0})u_3^{2,1} + 3I^{2,2}u_3^{0,3} = 2I^{1,2} - I^{3,0},$$

$$I^{2,2}u_3^{2,1} + 3I^{0,4}u_3^{0,3} = -I^{1,2},$$

from which

$$\begin{aligned} u_3^{2,1} &= 3[I^{0,4}(2I^{1,2} - I^{3,0}) + I^{1,2}I^{2,2}]/\Delta, \\ u_3^{0,3} &= -[I^{1,2}(4I^{2,2} + I^{4,0}) + I^{2,2}(2I^{1,2} - I^{3,0})]/\Delta, \end{aligned} \tag{24}$$

where

$$\Delta = 3[I^{0,4}(4I^{2,2} + I^{4,0}) - I^{2,2}I^{2,2}].$$

The torsional rigidity, from (21) with only $u_3^{2,1}$ and $u_3^{0,3}$, is

$$C = \mu[I^{0,2} + I^{2,0} + (I^{3,0} - 2I^{1,2})u_3^{2,1} + 3I^{1,2}u_3^{0,3}].$$

Then, employing the values (24) of $u_3^{2,1}$ and $u_3^{0,3}$ and noting that, from (20),

$$I^{2,0} = I^{0,2} = 3a^4\sqrt{3}/2,\ I^{1,2} = -I^{3,0} = 3a^5\sqrt{3}/5,\ I^{4,0} = I^{0,4} = 3I^{2,2} = 9a^6\sqrt{3}/5,$$

we find

$$C = 9\mu a^4\sqrt{3}/5. \tag{25}$$

If we insert the St. Venant torsion function (23) in the integrand in (4) and perform the integration over the triangular area, we find the same result as (25) for the torsional rigidity (Ref. [5], p. 266).

RECTANGLE

St. Venant's torsion function for the bar of rectangular section $x_1 = \pm a$, $x_2 = \pm b$ is (Ref. [4], p. 318)

$$\varphi = -x_1x_2 + 32b^2 \sum_{n=1,3\ldots}^{\infty} \frac{(-)^{(n-1)/2}\sinh\dfrac{n\pi x_1}{2b}\sin\dfrac{n\pi x_2}{2b}}{n^3\pi^3\cosh\dfrac{n\pi a}{2b}}.$$

As an approximation, we retain only $u_3^{1,1}$, $u_3^{1,3}$, $u_3^{3,1}$, $u_3^{3,3}$ in the power series expression (5) for φ. Then the equilibrium equations (16) are

$$\begin{aligned} &T_{31}^{0,1} + T_{32}^{0,1} = 0,\ T_{31}^{0,3} + 3T_{32}^{1,2} = 0,\\ &3T_{31}^{2,1} + T_{32}^{3,0} = 0,\ T_{31}^{2,3} + T_{32}^{3,2} = 0, \end{aligned} \tag{26}$$

and we find components of stress, from (18) and (19),

$$\begin{aligned} T_{31}^{0,1} &= \mu\tau(-I^{0,2} + I^{0,2}u_3^{1,1} + I^{0,4}u_3^{1,3} + 3I^{2,2}u_3^{3,1} + 3I^{2,4}u_3^{3,3}),\\ T_{32}^{1,0} &= \mu\tau(I^{2,0} + I^{2,0}u_3^{1,1} + 3I^{2,2}u_3^{1,3} + I^{4,0}u_3^{3,1} + 3I^{4,2}u_3^{3,3}),\\ T_{31}^{0,3} &= \mu\tau(-I^{0,4} + I^{0,4}u_3^{1,1} + I^{0,6}u_3^{1,3} + 3I^{2,4}u_3^{3,1} + 3I^{2,6}u_3^{3,3}),\\ T_{32}^{1,2} &= \mu\tau(I^{2,2} + I^{2,2}u_3^{1,1} + 3I^{2,4}u_3^{1,3} + I^{4,2}u_3^{3,1} + 3I^{4,4}u_3^{3,3}),\\ T_{31}^{2,1} &= \mu\tau(-I^{2,2} + I^{2,2}u_3^{1,1} + I^{2,4}u_3^{1,3} + 3I^{4,2}u_3^{3,1} + 3I^{4,4}u_3^{3,3}),\\ T_{32}^{3,0} &= \mu\tau(I^{4,0} + I^{4,0}u_3^{1,1} + 3I^{4,2}u_3^{1,3} + I^{6,0}u_3^{3,1} + 3I^{6,2}u_3^{3,3}),\\ T_{31}^{2,3} &= \mu\tau(-I^{2,4} + I^{2,4}u_3^{1,1} + I^{2,6}u_3^{1,3} + 3I^{4,4}u_3^{3,1} + 3I^{4,6}u_3^{3,3}),\\ T_{32}^{3,2} &= \mu\tau(I^{4,2} + I^{4,2}u_3^{1,1} + 3I^{4,4}u_3^{1,3} + I^{6,2}u_3^{3,1} + 3I^{6,4}u_3^{3,3}). \end{aligned} \tag{27}$$

Upon substituting (27) in (26), we obtain a set of four simultaneous equations for the four $u_3^{m,n}$:

$$\begin{aligned} a_{11}u_3^{1,1} + a_{21}u_3^{1,3} + a_{31}u_3^{3,1} + 3a_{41}u_3^{3,3} &= c_1,\\ a_{21}u_3^{1,1} + a_{22}u_3^{1,3} + a_{32}u_3^{3,1} + 3a_{42}u_3^{3,3} &= c_2,\\ a_{31}u_3^{1,1} + a_{32}u_3^{1,3} + a_{33}u_3^{3,1} + 3a_{43}u_3^{3,3} &= c_3,\\ a_{41}u_3^{1,1} + a_{42}u_3^{1,3} + a_{43}u_3^{3,1} + 3a_{44}u_3^{3,3} &= c_4, \end{aligned} \tag{28}$$

where, with $I^{m,n} = 4a^{m+1}b^{n+1}/(m+1)(n+1)$ and $k = b/a$,

$$a_{11} = I^{0,2} + I^{2,0} = 4a^3b(1+k^2)/3, \qquad a_{32} = 3(I^{2,4} + I^{4,2}) = 4a^5b^3(1+k^2)/5,$$
$$a_{21} = I^{0,4} + 3I^{2,2} = 4a^3b^3(5+3k^2)/15, \qquad a_{42} = 3I^{4,4} + I^{2,6} = 4a^5b^5(63+25k^2)/525,$$
$$a_{31} = I^{4,0} + 3I^{2,2} = 4a^5b(3+5k^2)/15, \qquad a_{33} = 9I^{4,2} + I^{6,0} = 4a^7b(5+21k^2)/35,$$
$$a_{41} = I^{2,4} + I^{4,2} = 4a^5b^3(1+k^2)/15, \qquad a_{43} = 3I^{4,4} + I^{6,2} = 4a^7b^3(25+63k^2)/525,$$
$$a_{22} = 9I^{2,4} + I^{0,6} = 4a^3b^5(21+5k^2)/35, \qquad a_{44} = I^{4,6} + I^{6,4} = 4a^7b^5(1+k^2)/35,$$
$$c_1 = I^{0,2} - I^{2,0} = -4a^3b(1-k^2)/3, \qquad c_3 = 3I^{2,2} - I^{4,0} = -4a^5b(3-5k^2)/15,$$
$$c_2 = I^{0,4} - 3I^{2,2} = -4a^3b^3(5-3k^2)/15, \qquad c_4 = I^{2,4} - I^{4,2} = -4a^5b^3(1-k^2)/15.$$

The solution of the four equations (28) is

$$\begin{aligned} u_3^{1,1} &= (k^4-1)(14+375k^2+14k^4)/\Delta, \\ u_3^{1,3} &= 35k^2(1-6k^2-7k^4)/b^2\Delta, \\ u_3^{3,1} &= 35k^2(7+6k^2-k^4)/a^2\Delta, \\ 3u_3^{3,3} &= 245k^2(k^4-1)/a^2b^2\Delta \end{aligned} \tag{29}$$

where

$$\Delta = 2(1+k^2)^2(7+121k^2+7k^4).$$

The torsional rigidity, from (21), is

$$C = \mu(I^{0,2} + I^{2,0} - c_1u_3^{1,1} - c_2u_3^{1,3} - c_3u_3^{3,1} - 3c_4u_3^{3,3}),$$

or

$$C = k_1\mu(2a)^3(2b), \tag{30}$$

where

$$12k_1 = 1 + k^2 - \frac{(1-k^2)^2(7+212k^2+7k^4)}{(1+k^2)(7+121k^2+7k^4)} + \frac{7k^2(13-25k^2-25k^4+13k^6)}{(1+k^2)^2(7+121k^2+7k^4)}.$$

Values of k_1 for various values of k are listed in Table 1 along with the corresponding values (Ref.

Table 1. Torsional rigidity coefficient in equation (30): k_1 (four terms of power series approximation); k_1' (St. Venant solution)

$k = b/a$	k_1	k_1'	b/a	k_1	k_1'
1·0	0·1407	0·1406	4	0·286	0·281
1·2	0·166	0·166	5	0·298	0·291
1·5	0·196	0·196	10	0·322	0·312
2·0	0·230	0·229	20	0·330	0·323
2·5	0·251	0·249	100	0·333	0·331
3·0	0·266	0·263	∞	1/3	1/3

[5], p. 277) calculated from St. Venant's solution. It may be seen, from (29), that for the square section ($k = 1$), $u_3^{1,1}$ and $u_3^{3,3}$ are zero; the approximation reduces to a two-term one with

$$u_3^{3,1} = -u_3^{1,3} = 7/18a^2, \quad k_1 = 19/135.$$

This simple approximation yields a torsional rigidity in error by only about 0·1 per cent. As k increases from unity, the error in k_1 increases to about 3 per cent around $k = 10$ and then diminishes to zero as k approaches infinity, at which limit $u_3^{1,3}$, $u_3^{3,1}$ and $u_3^{3,3}$, in (29), are zero and $u_3^{1,1}$ is unity, yielding the correct form for φ:

$$\varphi = \tau x_1 x_2$$

at that limit. The errors may be reduced by retaining more terms in the series, the next group being $u_3^{1,5}$, $u_3^{3,5}$, $u_3^{5,5}$, $u_3^{5,3}$, $u_3^{5,1}$.

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